



## Lectures 17/18 Derivatives and Graphs

When we have a picture of the graph of a function  $f(x)$ , we can make a picture of the derivative  $f'(x)$  using the slopes of the tangents to the graph of  $f$ . In this section we will think about using the derivative  $f'(x)$  and the second derivative  $f''(x)$  to help us reconstruct the graph of  $f(x)$ .

### Increasing Functions and Decreasing Functions

Recall that a function is (strictly) increasing on an interval  $[a, b]$  if for every pair of numbers  $x_1 < x_2$  in  $[a, b]$ , we have  $f(x_1) < f(x_2)$ . Similarly a function is (strictly) decreasing on an interval  $[a, b]$  if for every pair of numbers  $x_1 < x_2$  in  $[a, b]$ , we have  $f(x_1) > f(x_2)$ .

**Theorem**   Suppose that  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then

- If  $f'(x) > 0$  at each point  $x \in (a, b)$ , then  $f$  is increasing on  $[a, b]$
- If  $f'(x) < 0$  at each point  $x \in (a, b)$ , then  $f$  is decreasing on  $[a, b]$ .

**Proof** This follows from the Mean Value Theorem. Let  $x_1$  and  $x_2$  be any two points in  $[a, b]$  with  $x_1 < x_2$ . The Mean Value Theorem applied to  $f$  on  $[x_1, x_2]$  says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some  $c$  between  $x_1$  and  $x_2$ . If  $f'$  is positive on the interval  $(a, b)$ ,  $f'(c) > 0$  and since  $x_2 > x_1$ ,  $x_2 - x_1 > 0$ . Hence  $f(x_2) - f(x_1) > 0$  or  $f(x_2) > f(x_1)$  and  $f$  is increasing on the interval  $[a, b]$ . A similar argument shows that  $f$  is decreasing on the interval  $[a, b]$  if  $f'$  is negative on the interval  $(a, b)$ .

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### Finding intervals where $f$ is Increasing/Decreasing

To find the intervals where a function  $f$  is increasing or decreasing we must identify the intervals where  $f'$  is positive and negative. We first make a list of all points where the derivative (when it exists) might switch sign.

1. **Find the domain of  $f$ .** Make a list of the endpoints of the connected intervals in the domain. (For rational functions this is just a list of isolated points not in the domain)
2. **First find the critical points of  $f$ .** Recall that these are the points in the domain of  $f$  where  $f'(x) = 0$  or  $f'(x)$  does not exist.
3. **Check if  $f'$  is continuous on the intervals between the points** on the list from 1 and 2. Note that if  $f'$  is continuous on  $(a, b)$ , with no zeros in the interval, then by the Intermediate Value theorem applied to  $f'$ , we must have that  $f'$  is positive everywhere on  $(a, b)$  or negative everywhere on  $(a, b)$ .
4. **Check the sign of  $f'$  on the above intervals** by checking the value of  $f'$  at a point in the interval or by checking the sign of the factors of  $f'$  if it is a rational function.
5.  **$f$  is increasing on the intervals where  $f' > 0$  and decreasing on the intervals where  $f' < 0$ .**

**Example** Find where the function  $f(x) = 2x^3 - 3x^2 - 12x + 4$  is increasing and where  $f$  is decreasing.

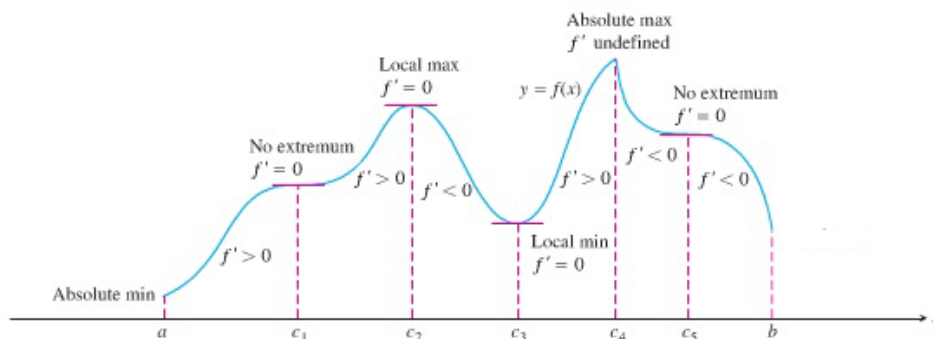
### First Derivative Test for local extrema (maxima and minima).

**First Derivative Test**   Suppose

- $c$  is a critical point of a continuous function  $f$  and
- $f$  is differentiable at every point in some interval containing  $c$  except possibly at  $c$  itself.

Then moving across this interval from left to right we get

1. If  $f'$  changes from negative to positive at  $c$ , then  $f$  has a local minimum at  $c$ ;
2. If  $f'$  changes from positive to negative at  $c$ , then  $f$  has a local maximum at  $c$ ;
3. If  $f'$  does not change sign at  $c$  (that is  $f'$  is positive on both sides of  $c$  or negative on both sides of  $c$ ), then  $f$  has no local extrema at  $c$ .



**Example** Let  $f(x) = 2x^3 - 3x^2 - 12x + 4$  as in the previous example. Classify the critical points as either local maxima, local minima or neither.

**Example** Find the critical points of

$$f(x) = x^{1/3}(x^2 - 4).$$

Identify the intervals on which  $f$  is increasing and decreasing. Find the function's local maxima and minima. Draw a rough sketch of the graph of the function.

Finding the critical points and determining the sign of  $f'$  are skills already mastered, so we include this part of the solution in the notes so that we have time to focus on the interpretation of the results.

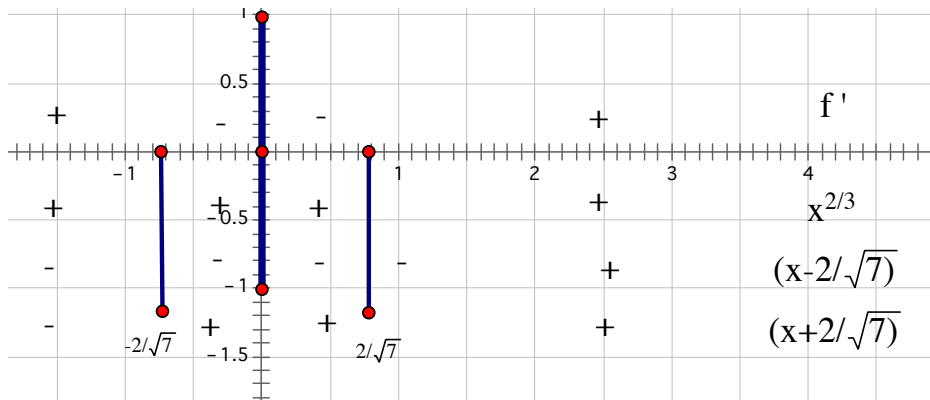
$$f'(x) = (x^2 - 4)\frac{1}{3x^{2/3}} + x^{1/3}(2x) = \frac{(x^2 - 4) + 3x(2x)}{3x^{2/3}} = \frac{x^2 + 6x^2 - 4}{3x^{2/3}} = \frac{7x^2 - 4}{3x^{2/3}}.$$

Critical points:

$$f'(x) \text{ D.N.E. if } \boxed{x = 0} \text{ and } f'(x) = 0 \text{ if } 7x^2 - 4 = 0 \text{ or } x^2 = \frac{4}{7} \text{ or } \boxed{x = \pm \frac{2}{\sqrt{7}}}.$$

$$f'(x) = \frac{7x^2 - 4}{3x^{2/3}} = \frac{7(x^2 - \frac{4}{7})}{3x^{2/3}} = \frac{7(x - \frac{2}{\sqrt{7}})(x + \frac{2}{\sqrt{7}})}{3x^{2/3}} = \frac{7}{3} \frac{(x - \frac{2}{\sqrt{7}})(x + \frac{2}{\sqrt{7}})}{x^{2/3}}.$$

We can analyze the sign of the derivative by multiplying the signs of its factors.



The function  $f$  is increasing on

$f$  is decreasing on

Using the first derivative test, we get  
a local maximum at

a local minimum at

Neither a maximum nor a minimum at

**Example** Find the local maxima and minima of the function  $g(x) = 1 - \cos^2 x$  on the interval  $[-3\pi/4, 3\pi/4]$ .

$$g'(x) = -2(\cos x)(-\sin x) = 2 \cos x \sin x = \sin 2x.$$

Critical Points:

$g'(x) > 0$  when

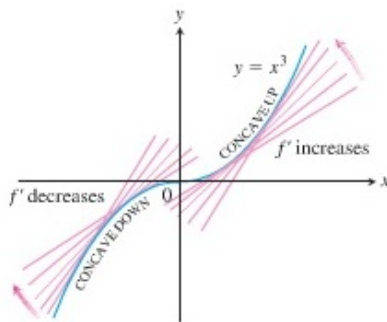
$g'(x) < 0$  when

Local Max:

Local Min:

### The Second Derivative and the graph of a function.

**Definition (Concavity)** If the graph of  $f$  lies above all of its tangents on an interval  $I$ , we say that the graph of  $f$  is **Concave up** on  $I$ . If the graph of  $f$  lies below all of its tangents on an interval  $I$ , we say that the graph of  $f$  is **Concave down** on  $I$ .



We see that  $f(x) = x^3$  is concave up on the interval  $(0, \infty)$  and concave down on the interval  $(-\infty, 0)$ .

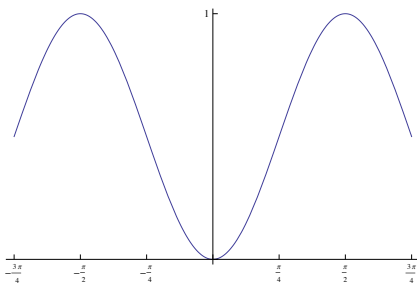
We also see that a function  $f$  is concave up if the derivative  $f'$  is increasing and concave down if the derivative  $f'$  is decreasing. this gives us the concavity test using the second derivative:

#### Concavity Test



1. If  $f''(x) > 0$  for all  $x$  in an interval  $I$ , then the graph of  $f$  is concave upward on  $I$ .
2. If  $f''(x) < 0$  for all  $x$  in an interval  $I$ , then the graph of  $f$  is concave downward on  $I$ .

**Example** On which intervals is the function  $g(x) = 1 - \cos^2 x$  for  $-3\pi/4 \leq x \leq 3\pi/4$  concave up and concave down (see graph below) ?



$$g'(x) = \sin(2x), \quad g''(x) =$$

$$g''(x) > 0 \text{ if}$$

$$g''(x) < 0 \text{ if}$$

Graph is concave up on

Graph is concave down on

Note that there are some points on the curve above where the graph switches from being concave up to being concave down and vice versa.

**Definition** A point  $P$  on a curve  $y = f(x)$  is called an **inflection point** if  $f$  is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at that point.

**Example** Find the points of inflection on the curve  $y = 1 - \cos^2 x$  on the interval  $[-3\pi/4, 3\pi/4]$ .

## Second Derivative Test for local maxima and minima



We can also use the second derivative to classify the local extrema:

**Second Derivative test** Suppose that  $f''(x)$  is continuous on an open interval containing  $c$ .

- If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local minimum at  $c$ .
- If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local maximum at  $c$ .
- If  $f'(c) = 0$  and  $f''(c) = 0$ , then the test is inconclusive.

**Example** Consider the function  $f(x) = x^4 - 4x^3 + 10$ .

(a) Identify the critical points.

The critical points occur at points in the domain of  $f$  where  $f'(x)$  does not exist and where  $f'(x) = 0$ .

$f'(x) = 4x^3 - 12x^2$ . It exists everywhere because  $f(x)$  is a polynomial function.

$f'(x) = 0$  if  $4x^3 - 12x^2 = 0$ . This happens if  $4x^2(x - 3) = 0$  or  $x = 0, x = 3$ .

Thus the critical points are  $x = 0, 3$ .

(b) Use the second derivative test to check for local extrema.

(c) Where is the curve Increasing/Decreasing?

(d) Where is the curve concave up/concave down? Where are the points of inflection?

(e) Draw a rough sketch of the graph below

**Example** Sketch a smooth connected curve  $y = f(x)$  with;

$$f(-2) = 8, \quad f(0) = 4, \quad f(2) = 0.$$

$$f'(x) > 0 \text{ for } |x| > 2, \quad f'(2) = f'(-2) = 0, \quad f'(x) < 0 \text{ for } |x| < 2$$

$$f''(x) < 0 \text{ for } x < 0, \quad f''(x) > 0 \text{ for } x > 0.$$